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A Short Existence Proof for Correlation Dimension

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The Grassberger Hentschel–Procaccia correlation dimension has been put on a rigorous basis by Pesin and Tempelman. We simplify their proof that this dimension is given in terms of the measure of neighborhoods of the diagonal.

KEY WORDS: Correlation dimension; ergodic measure; neighborhood of diagonal.

Let (X, ρ) be a separable metric space. Suppose that μ is an ergodic probability measure for the continuous map $f: X \to X$. The r neighbourhood of the diagonal in $X \times X$ is denoted by S_r. That is $S_r := \{(x, y) \in X \times X: \rho(x, y) \leqslant r\}$. The function $\varphi(r) = \nu(S_r)$ is monotone increasing where ν is the product measure $\mu \times \mu$. For $x \in X$ and $n \in \mathbf{N}$ we let $C(x, n, r)$ denote $1/n^2 \# \{(i, j): (f^i(x), f^j(x)) \in S_r, 0 \leqslant i, j < n\}$, the proportion of pairs of points in part of the orbit that are closer than r. Roughly speaking, if, for μ almost every x, for large n and small r, we have $C(x, n, r) \sim r^\alpha$ then α is called the correlation dimension[3] of the measure μ. To give a precise definition of the correlation dimension it is fundamental to prove the following theorem, as was done for invertible f by Pesin[1] and, in a more general context, by Pesin and Tempelman.[2]

Theorem 1. There is a set $Y \subset X$ of full μ-measure such that for each $x \in Y$

$$C(x, n, r) \to \varphi(r) \qquad \text{as} \quad n \to \infty \tag{1}$$

provided φ is continuous at r.

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Hence the correlation dimension of μ, $\liminf_{r\to 0} \log \varphi(r)/\log r$ can be estimated from a long μ-typical trajectory of f. Our aim here is to give a short proof of this theorem.

Proof. For each $m \in \mathbf{N}$ take a finite partition of X, denoted $\mathscr{A}^m := \{A_j^m : 0 \leqslant j \leqslant M(m)\}$, such that $\mu(A_0^m) \leqslant 2^{-m}$ and $\operatorname{diam}(A_j^m) \leqslant 2^{-m}$ for $0 < j \leqslant M(m)$. Since X is separable and $\mu(X) < \infty$ such a partition obviously exists. Fix $m \in \mathbf{N}$ and $r > 0$. Let $\mathscr{C}' := \{C \in \mathscr{A}^m \times \mathscr{A}^m : C \subset S_r\}$ and $\mathscr{C}'' := \{C \in \mathscr{A}^m \times \mathscr{A}^m : C \cap S_r \neq \varnothing\}$. The next two inequalities follow immediately from the definitions:

$$\sum_{C \in \mathscr{C}'} \frac{1}{n^2} \sharp \{(i, j) : (f^i(x), f^j(x)) \in C, 0 \leqslant i, j < n\} \leqslant C(x, n, r)$$
$$\leqslant \sum_{C \in \mathscr{C}''} \frac{1}{n^2} \sharp \{(i, j) : (f^i(x), f^j(x)) \in C, 0 \leqslant i, j < n\}. \tag{2}$$

Further,

$$S_{r-2^{-m+1}} \setminus ((A_0^m \times X) \cup (X \times A_0^m))$$
$$\subset \bigcup_{C \in \mathscr{C}'} C \subset \bigcup_{C \in \mathscr{C}''} C \subset S_{r+2^{-m+1}} \cup ((A_0^m \times X) \cup (X \times A_0^m)). \tag{3}$$

By the Birkhoff ergodic theorem we are able to choose $Y \subset X$ with $\mu(Y) = 1$ such that

$$\forall x \in Y, \quad \forall m \in \mathbf{N} \qquad \text{and}$$
$$\forall A \in \mathscr{A}^m : \frac{1}{n} \sharp \{i \in [0, n) : f^i(x) \in A\} \to \mu(A) \qquad \text{as} \quad n \to \infty$$

Fix $x \in Y$. For each m, we choose N such that

$$\forall n > N, \ \forall A \in \mathscr{A}^m : \left| \frac{1}{n} \sharp \{i \in [0, n) : f^i(x) \in A\} - \mu(A) \right| < \frac{2^{-m-1}}{(M(m))^2}$$

Then, for each $C = A \times A'$, with $A, A' \in \mathscr{A}^m$,

$$\left| \frac{1}{n^2} \sharp \{(i, j) : (f^i(x), f^j(x)) \in C, 0 \leqslant i, j < n\} - \nu(C) \right|$$
$$= \left| \frac{1}{n} \sharp \{i \in [0, n) : f^i(x) \in A\} \cdot \frac{1}{n} \sharp \{j \in [0, n) : f^j(x) \in A'\} - \mu(A)\,\mu(A') \right|$$
$$< \frac{2^{-m}}{(M(m))^2}. \tag{4}$$

Thus from this and (2) we obtain

$$\sum_{C\in\mathscr{C}'}\left(\nu(C)-\frac{2^{-m}}{(M(m))^2}\right)<C(x,n,r)<\sum_{C\in\mathscr{C}''}\left(\nu(C)+\frac{2^{-m}}{(M(m))^2}\right). \quad (5)$$

Using this and (3) we obtain that

$$\forall n>N\colon \nu(S_{r-2^{-m+1}})-2^{-m}-2\cdot 2^{-m} < C(x,n,r)<\nu(S_{r+2^{-m+1}})+2^{-m}+2\cdot 2^{-m}. \quad (6)$$

Let $m\to\infty$. Since $\varphi(r)=\nu(S_r)$ is continuous at r we immediately obtain the statement of the theorem.

We remark that in our proof above we have never used the property of metric spaces that $\rho(x, y)=0$ implies that $x=y$. So our proof gives more than Theorem 1. What we proved in fact is that the statement of Theorem 1 holds if (X, ρ) is a separable pseudo-metric space. ($\rho\colon X\times X\to\mathbf{R}^+$ is called a pseudo-metric if $\rho(x,z)\leqslant\rho(x,y)+\rho(y,z)$ and $\rho(x,y)=\rho(y,x)$.)

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